Cobb-Douglas Based Firm Production Model in Fuzzy Environment and its Solution Procedure by Geometric Programming Technique

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Abstract

In this paper we consider Cobb-Douglas production function based models in a firm and its solution technique under fuzzy environment. A firm uses finite inputs such as labor, capital, coal, iron etc. to produce a single output. In fact production function of a firm is nothing but the existence of maximum output corresponding to any combination of inputs. It is correctly said that the firm is in a competitive situation if it can buy and sell in any quantities of exogenously given prices that are independent of its production decisions. In other words, the competitive firm is a price taker. But in real life decision making, the coefficients and/or objectives are not usually crisp quantities. These are better represented by fuzzy quantities. In this paper, we have considered a model such that a firm behaves so as to maximize the revenue under limited total expenditure cost in fuzzy environment. There are different techniques to solve fuzzy optimization models. The model is solved by applying geometric programming technique. Geometric programming has many advantages over other approaches. Illustrative numerical examples are provided to demonstrate the feasibility and efficiency of proposed models. Conclusions are drawn at last.

Keywords: Cobb-Douglas production function; firm production model; geometric programming technique; fuzzy optimization; fuzzy decision making; fuzzy geometric programming; fuzzy parametric programming.

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1. Introduction and preliminaries

The Cobb-Douglas production function is widely used to represent the relationship of an output to inputs. Knut Wicksell (1851-1926) had proposed the production function, and it is tasted against statistical evidence by Charles Cobb and Paul Douglas in 1928. In 1928 Charles Cobb and Paul Douglas published a study in which they modelled the growth of American economy during the period 1899-1922. They considered a simplified view of the economy in which production output is determined by the amount of labor involved and the

amount of capital invested. While there are many other factors affecting the production output, their model was remarkably accurate. The production function used in that model is

$$P(L,K) = aL^{\alpha}K^{\beta}$$

where:

P: Total monetary value of all production (goods produced in a year)
L: Total labor input (number of person –hours worked in a year)
K: Total capital input (the monetary worth of all machinery, equipment and buildings)
a: Total factor productivity,
α, β: The output elasticities of labor and capital respectively.

Available technology determines these values and they are usually constants. It may be noted that output elasticity measures the responsive of output to a change in levels of either labor or capital used in production e.g. for α =0.25, single 1% increase in labor would lead to approximately 0.25% increase in output. Moreover for normalized case: $\alpha + \beta = 1$, the production function has constant returns to scale. Hence one 10% increase in both L and K increases P by 10%. Here returns to scale is a technical property of production that examines changes in output subsequent to proportional change in all inputs, here all inputs increase by a constant factor. Again for $\alpha + \beta < 1$ returns to scale are decreasing and for $\alpha + \beta > 1$ returns to scale are increasing. In fact in the scenario of perfect competition, α and β are labor's and capitals' share of output. Analogous to Shivanian, Keshtkar and Khorram (2013), inference is viewed as a process of propagation of elastic constraints.

On the other hand, there are several modifications and changes in classical set theory. One of these that brought a paradigm shift in mathematics is the concept of fuzzy set theory, introduced by Lotfi Asker Zadeh (1965). And according to Bellman and Zadeh (1970), fuzzy set is a better representation of real life situations than classical crisp set. Hence in production planning, Cobb-Douglas production function may also be considered under fuzzy environment. As Guney and Oz (2010) suggested, geometric programming technique can be applied in Cobb-Douglas based firm production model under fuzzy environment. As Cao (2010) mentioned, it is well known that geometric programming technique provides us with a systematic approach for solving a class of non-linear optimization problems by finding the optimal value of objective function and then the optimal values of decision variables are obtained.

This paper is arranged as follows. In sect. 2, methodology is discussed in detail by different approaches one after another. Next in Sect. 3, numerical examples are given using different fuzzy optimization techniques. Then the results obtained by using those different techniques are presented in a table. Finally, conclusions are drawn in Sect 4.

2. Cobb-Douglas Based Firm Production Model in Fuzzy Environment

In this paper, we have considered a firm that uses n inputs (e.g. labor, capital, coal, iron …) to produce a single output q. Thus the firm production function is represented by $q = f(x_1 \dots x_n)$. This function gives output as a function of its inputs i.e.

$$f(x_1 \dots x_n) = \prod_{i=1}^{n} a x_i^{\alpha_i}$$

where $\alpha_i \ (i=1\ldots n)$ are the output elasticities of the input components $x_i \ (i=1\ldots n)$. So total revenue amount is

$$pq = \prod_{i=1}^{n} pax_i^{\alpha_i}$$

Again if $r_i \ (i=1\ldots n)$ are the prices of the inputs $x_i \ (i=1\ldots n)$, then total expenditure cost is given by

$$C\left(x_1 \ldots x_n\right) = \sum_{i=1}^{n} r_i x_i$$

Here our consideration is to maximize total revenue under total limited expenditure cost. Hence in this case, according to Liu (2006), Cobb-Douglas based firm production model under crisp environment is as follows

$$\begin{aligned} &\text{Maximize } R\left(x_1 \ldots x_n\right) = \prod_{i=1}^{n} pax_i^{\alpha_i} \\ &\text{subject to } C\left(x_1 \ldots x_n\right) \equiv \sum_{i=1}^{n} r_i x_i \le c,\ x_i > 0, i=1\ldots n. \end{aligned} \tag{2.1}$$

Using the method described in Duffin et al (1967), geometric programming technique can be applied to solve this model.

Now suppose we consider Cobb Douglas production model under fuzzy environment, with constraint $\tilde{C} = \left\{\left(C\left(x_1 \ldots x_n\right), \mu_{\tilde{C}}\left(C\left(x_1 \ldots x_n\right)\right)\right)\right\}$ being fuzzy, as follows

$$\begin{aligned} &\text{Maximize } R\left(x_1 \ldots x_n\right) = \prod_{i=1}^{n} pax_i^{\alpha_i} \\ &\text{subject to } C\left(x_1 \ldots x_n\right) \equiv \sum_{i=1}^{n} r_i x_i \prec c \\ &\text{with maximum allowable tolerances } c_0 \\ &x_i > 0, i=1\ldots n. \end{aligned} \tag{2.3}$$

Here membership function of fuzzy constraint $\tilde{C} = \left\{\left(C\left(x_1 \ldots x_n\right), \mu_{\tilde{C}}\left(C\left(x_1 \ldots x_n\right)\right)\right)\right\}$ is

$$\mu_{\tilde{C}}\left(C\left(x_1 \ldots x_n\right)\right) = \begin{cases} 0 & ,\text{if } C\left(x_1 \ldots x_n\right) \ge c + c_0 \\ \dfrac{c + c_0 - C\left(x_1 \ldots x_n\right)}{c_0} & ,\text{if } c \le C\left(x_1 \ldots x_n\right) \le c + c_0 \\ 1 & ,\text{if } C\left(x_1 \ldots x_n\right) \le c \end{cases}$$

Next we plan to apply different fuzzy optimization techniques on model (2.3) one after another.

Method 2.1. Verdegay's approach

According to Verdegay's approach (1982) on fuzzy optimization, model (2.3) reduces to following parametric optimization model as

$$\text{Maximize } R\left(x_1 \ldots x_n\right)=\prod_{i=1}^{n} pa x_i^{\alpha_i}$$
$$\text{subject to } \sum_{i=1}^{n} r_i x_i \le c+(1-\beta)c_0, \beta\in[0,1],\ x_i>0,\ i=1\ldots n.$$

In PGPP form, the above model becomes

$$\begin{aligned}&\text{Minimize } \prod_{i=1}^{n} \frac{1}{pa} x_i^{-\alpha_i}\\ &\text{subject to } \frac{1}{c+(1-\beta)c_0}\sum_{i=1}^{n} r_i x_i \le 1\\ &x_i>0, i=1\ldots n.\end{aligned} \tag{2.4}$$

Model (2.4) is also a posynomial PGPP with DD being zero. Therefore its DGPP is as follows

$$\text{Maximize } d\left(\delta_{01},\delta_{11},\delta_{12}\ldots\delta_{1n}\right)=$$
$$\left(\frac{1}{pa\delta_{01}}\right)^{\delta_{01}} \prod_{i=1}^{n}\left(\frac{r_i}{(c+(1-\beta)c_0)\delta_{1i}}\right)^{\delta_{1i}} \left(\sum_{i=1}^{n}\delta_{1i}\right)^{\sum_{i=1}^{n}\delta_{1i}}$$
$$\text{subject to } \delta_{01}=1, -\alpha_i\delta_{01}+\delta_{1i}=0,\ \delta_{1i}>0\ \forall\ i=1\ldots n.$$

The optimal solution of this problem is obtained as $\delta_{01}^*=1,\ \delta_{1i}^*=\alpha_i$ for $i=1\ldots n$. Again from the primal dual relations $\prod_{i=1}^{n}\frac{1}{pa}x_i^{-\alpha_i}=\delta_{01}^*\ d^*\left(\delta_{01}^*,\delta_{11}^*,\delta_{12}^*\ldots\delta_{1n}^*\right)$ and $\frac{r_i x_i}{c+(1-\beta)c_0}=\frac{\delta_{1i}^*}{\sum_{i=1}^{n}\delta_{1i}^*}\ \forall\ i=1\ldots n$.

Hence optimal inputs are $x_i^*(\beta)=\frac{(c+(1-\beta)c_0)\alpha_i}{r_i\sum_{i=1}^{n}\alpha_i}\ \forall i=1\ldots n$ and optimal revenue is given by

$$R^*\left(x_1^*\ldots x_n^*;\beta\right)=pa\left(\frac{c+(1-\beta)c_0}{\sum_{i=1}^{n}\alpha_i}\right)^{\sum_{i=1}^{n}\alpha_i}\prod_{i=1}^{n}\left(\frac{\alpha_i}{r_i}\right)^{\alpha_i}.$$

Method 2.2. Werner's approach

Here the reduced model is first solved without tolerance by geometric programming technique and then solved with tolerance by geometric programming technique and finally fuzzy non-linear programming problem is obtained as follows

$$\begin{aligned}&\textit{Maximize } R\left(x_1\ldots x_n\right)=\prod_{i=1}^{n} pa x_i^{\alpha_i}\in[R_0,R_1]\\ &\textit{subject to}\\ &C\left(x_1\ldots x_n\right)\equiv\sum_{i=1}^{n} r_i x_i\le c \textit{ with } \max\textit{imum allowable tolerances } c_0,\\ &x_i>0,\ i=1\ldots n.\end{aligned}$$

here the revenue without tolerance and with tolerance being R_0 *and* R_1 respectively. Therefore the task is to find

$$
\begin{aligned}
& x_i,\ i = 1 \ldots n \\
& \textit{subject}\ \text{to} \\
& R\left(x_1 \ldots x_n\right) \equiv \prod_{i=1}^{n} p a x_i^{\alpha_i} \succ R_1 \textit{ with maximum allowable tolerance } (\mathrm{R}_1 - R_0), \\
& C\left(x_1 \ldots x_n\right) \equiv \sum_{i=1}^{n} r_i x_i \prec c \textit{ with maximum allowable tolerance } c_0, \\
& x_i > 0,\ i = 1 \ldots n.
\end{aligned}
$$

The objective function has fuzzy goal and is given by $\tilde{R} = \{R(x_1 \ldots x_n), \mu_{\tilde{R}}(R(x_1 \ldots x_n))\}$ with linear membership function as follows

$$
\mu_{\tilde{R}}(R(x_1 \ldots x_n)) = \begin{cases} 0 & , \text{if } R(x_1 \ldots x_n) \le R_0 \\ \dfrac{R(x_1 \ldots x_n) - R_0}{R_1 - R_0} & , \text{if } R_0 \le R(x_1 \ldots x_n) \le R_1 \\ 1 & , \text{if } R(x_1 \ldots x_n) \ge R_1 \end{cases}
$$

And constraint is also fuzzy given by $\tilde{C} = \left\{\left(C(x_1 \ldots x_n), \mu_{\tilde{C}}(C(x_1 \ldots x_n))\right)\right\}$. Here task is to find x_i so as to maximize the minimum of $\mu_{\tilde{R}}\left(R\left(x_1 \ldots x_n\right)\right) \textit{and}\, \mu_{\tilde{C}}\left(C\left(x_1 \ldots x_n\right)\right) \textit{and } x_i > 0,\ i = 1 \ldots n.$

Method 2.3. Zimmermann's approach (1975)

Here the model (2.3) is solved by using max-min operator as follows. Suppose $\beta = \left(\text{minimum}\left\{\mu_{\tilde{R}}(R\left(x_1 \ldots x_n\right)), \mu_{\tilde{C}}\left(C\left(x_1 \ldots x_n\right)\right)\right\}\right)$. Then we have the single objective optimization problem as follows

$$
\begin{aligned}
& \text{Maximize } \beta \\
& \text{subject to} \\
& \frac{c + c_0 - C\left(x_1 \ldots x_n\right)}{c_0} \ge \beta, \frac{R\left(x_1 \ldots x_n\right) - R_0}{R_1 - R_0} \ge \beta, \\
& x_i > 0 \forall i = 1 \ldots n.
\end{aligned}
$$

Using reciprocal, we obtain a posynomial geometric programming problem with DD being 2. To solve it using geometric programming technique, the dual is as follows

$$\text{Maximize } d\left(\delta_{01},\delta_{11},\delta_{12}\ldots\delta_{1n+1},\delta_{21},\delta_{22}\right) \tag{2.5}$$

$$= \left(\frac{1}{\delta_{01}}\right)^{\delta_{01}} \prod_{i=1}^{n}\left(\frac{r_i}{(c+c_0)\delta_{1i}}\right)^{\delta_{1i}} \left(\frac{c_0}{(c+c_0)\delta_{1n+1}}\right)^{\delta_{1n+1}}$$

$$\left(\frac{R_0}{pa\delta_{21}}\right)^{\delta_{21}} \left(\frac{R_1-R_0}{pa\delta_{22}}\right)^{\delta_{22}} \left(\sum_{i=1}^{n+1}\delta_{1i}\right)^{\sum_{i=1}^{n+1}\delta_{1i}} \left(\sum_{i=1}^{2}\delta_{2i}\right)^{\sum_{i=1}^{2}\delta_{2i}}$$

subject to

$$\delta_{01}=1,\ -\delta_{01}+\delta_{1n+1}+\delta_{22}=0,$$

$$\delta_{1i}-\alpha_i\delta_{21}-\alpha_i\delta_{22}=0,\ \forall\ i=1\ldots n.$$

The optimal solution of model (2.5) is given by $\delta_{01}^{*}=1, \delta_{1n+1}=(1-\delta_{22}), \delta_{1i}=\alpha_i(\delta_{21}+\delta_{22})\ \forall\, i=1..n$. Now substituting δ_{01}^{*}, δ_{1i} (for $i=1\ldots n$), δ_{1n+1} in model (2.5), the dual function is obtained as follows

$$\textit{Maximize } d(\delta_{21},\delta_{22}) = \prod_{i=1}^{n}\left(\frac{r_i}{(c+c_0)(\delta_{21}+\delta_{22})\alpha_i}\right)^{(\delta_{21}+\delta_{22})\alpha_i} \left(\frac{c_0}{(c+c_0)(1-\delta_{22})}\right)^{(1-\delta_{22})} \tag{2.6}$$

$$\left(\frac{R_0}{pa\delta_{21}}\right)^{\delta_{21}} \left(\frac{R_1-R_0}{pa\delta_{22}}\right)^{\delta_{22}} (\delta_{21}+\delta_{22})^{(\delta_{21}+\delta_{22})}$$

$$\left[\left(\sum_{i=1}^{n}\alpha_i\right)(\delta_{21}+\delta_{22})+(1-\delta_{22})\right]^{\left[\left(\sum_{i=1}^{n}\alpha_i\right)(\delta_{21}+\delta_{22})+(1-\delta_{22})\right]}$$

To find the optimal δ_{21},δ_{22}, we have to maximize dual objective $d(\delta_{21},\delta_{22})$. We take logarithm on both sides of model (2.6) and differentiate partially with respect to δ_{21},δ_{22}.

Next on equating these to zero, we have $\frac{\partial}{\partial\delta_{21}}\left[\log\{d(\delta_{21},\delta_{22})\}\right]=0$ and $\frac{\partial}{\partial\delta_{22}}\left[\log\{d(\delta_{21},\delta_{22})\}\right]=0$ i.e.

$$\sum_{i=1}^{n}\alpha_i\log\left(\frac{r_i}{(c+c_0)(\delta_{21}+\delta_{22})\alpha_i}\right)+\log\left(\frac{R_0}{pa\delta_{21}}\right)+\left(\sum_{i=1}^{n}\alpha_i\right)\log\left[\left(\sum_{i=1}^{n}\alpha_i\right)(\delta_{21}+\delta_{22})+(1-\delta_{22})\right]+\log(\delta_{21}+\delta_{22})=0 \text{ and}$$

$$\sum_{i=1}^{n}\alpha_i\log\left(\frac{r_i}{(c+c_0)(\delta_{21}+\delta_{22})\alpha_i}\right)-\log\left(\frac{c_0}{(c+c_0)(1-\delta_{22})}\right)+\log\left(\frac{R_1-R_0}{pa\delta_{22}}\right)+\left(\sum_{i=1}^{n}\alpha_i-1\right)\log\left(\left(\sum_{i=1}^{n}\alpha_i\right)(\delta_{21}+\delta_{22})+(1-\delta_{22})\right)+\log(\delta_{21}+\delta_{22})=0$$

Again we have $$\frac{\partial^2}{\partial\delta_{21}^2}\left[\log\{d(\delta_{21},\delta_{22})\}\right]=-\frac{\delta_{22}}{\delta_{21}(\delta_{21}+\delta_{22})}-\frac{(1-\delta_{22})\left(\sum_{i=1}^{n}\alpha_i\right)}{(\delta_{21}+\delta_{22})\left(\left(\sum_{i=1}^{n}\alpha_i\right)(\delta_{21}+\delta_{22})+(1-\delta_{22})\right)}<0$$

and $$\frac{\partial^2}{\partial\delta_{22}^2}\left[\log\{d(\delta_{21},\delta_{22})\}\right]=-\frac{\delta_{21}+\delta_{22}\left(\sum_{i=1}^{n}\alpha_i\right)}{\delta_{22}(\delta_{21}+\delta_{22})}-\frac{(1-\delta_{22})\left(\sum_{i=1}^{n}\alpha_i\right)\left(2-\left(\sum_{i=1}^{n}\alpha_i\right)\right)+\left(\sum_{i=1}^{n}\alpha_i\right)(\delta_{21}+\delta_{22})}{(1-\delta_{22})\left(\left(\sum_{i=1}^{n}\alpha_i\right)(\delta_{21}+\delta_{22})+(1-\delta_{22})\right)}<0$$

and $\frac{\partial^2}{\partial\delta_{21}\partial\delta_{22}}\left[\log\left\{d\left(\delta_{21},\delta_{22}\right)\right\}\right]=\frac{\partial^2}{\partial\delta_{22}\partial\delta_{21}}\left[\log\left\{d\left(\delta_{21},\delta_{22}\right)\right\}\right]=\frac{\left(1-\delta_{22}\right)\left(1-\sum_{i=1}^{n}\alpha_i\right)}{\left(\delta_{21}+\delta_{22}\right)\left(\left(\sum_{i=1}^{n}\alpha_i\right)\left(\delta_{21}+\delta_{22}\right)+\left(1-\delta_{22}\right)\right)}$

and may be observed that $\frac{\partial^2}{\partial\delta_{21}^2}\left[\log\left\{d\left(\delta_{21},\delta_{22}\right)\right\}\right].\frac{\partial^2}{\partial\delta_{22}^2}\left[\log\left\{d\left(\delta_{21},\delta_{22}\right)\right\}\right]-\left(\frac{\partial^2}{\partial\delta_{21}\partial\delta_{22}}\left[\log\left\{d\left(\delta_{21},\delta_{22}\right)\right\}\right]\right)^2>0.$

Method 2.4. Sakawa's method (2013)

Model (2.3) is now solved by using Sakawa's method. We suppose that $\mu_{\tilde{R}}\left(R\left(x_1\ldots x_n\right)\right)=\text{minimum}\left\{\mu_{\tilde{R}}(R\left(x_1\ldots x_n\right)),\mu_{\tilde{C}}\left(C\left(x_1\ldots x_n\right)\right)\right\}$. Therefore the model reduces to

$$\begin{aligned}&\text{Maximize } \mu_{\tilde{R}}\left(R\left(x_1\ldots x_n\right)\right)\\&\text{subject to}\\&\mu_{\tilde{C}}\left(C\left(x_1\ldots x_n\right)\right)\geq\mu_{\tilde{R}}\left(R\left(x_1\ldots x_n\right)\right),\\&x_i>0,\ \text{for } i=1\ldots n,\\&\mu_{\tilde{R}}\left(R\left(x_1\ldots x_n\right)\right),\mu_{\tilde{C}}\left(C\left(x_1\ldots x_n\right)\right)\in[0,1].\end{aligned}$$

i. e. we have

$$\begin{aligned}&\text{Maximize } \mu_{\tilde{R}'}\left(R\left(x_1\ldots x_n\right)\right)=\frac{\prod_{i=1}^{n}pax_i^{\alpha_i}}{R_1-R_0}\\&\text{subject to}\\&\frac{R_1-R_0}{R_1c_0+\left(R_1-R_0\right)c}\sum_{i=1}^{n}r_ix_i+\frac{pac_0}{R_1c_0+\left(R_1-R_0\right)c}\prod_{i=1}^{n}x_i^{\alpha_i}\leq 1\end{aligned}\tag{2.7}$$

here $\mu_{\tilde{R}}\left(R\left(x_1\ldots x_n\right)\right)=\mu_{\tilde{R}'}\left(R\left(x_1\ldots x_n\right)\right)-\frac{R_0}{R_1-R_0}$. To solve model (2.7) in geometric programming technique, we rewrite the problem as follows:

$$\begin{aligned}&\text{Minimize } \theta\prod_{i=1}^{n}x_i^{-\alpha_i}\\&\text{subject to } \theta_1\sum_{i=1}^{n}r_ix_i+\theta_2\prod_{i=1}^{n}x_i^{\alpha_i}\leq 1,\ x_i>0,\ i=1\ldots n.\end{aligned}\tag{2.8}$$

where $\theta=\frac{R_1-R_0}{pa}$, $\theta_1=\frac{R_1-R_0}{R_1c_0+\left(R_1-R_0\right)c}$, $\theta_2=\frac{pac_0}{R_1c_0+\left(R_1-R_0\right)c}$. Model (2.8) is a posynomial PGPP with DD being 1. Its DGPP form is given by

$$\text{Maximize } d\left(\delta_{01},\delta_{11},\delta_{12}\ldots\delta_{1n+1}\right)=\left(\frac{\theta}{\delta_{01}}\right)^{\delta_{01}}\prod_{i=1}^{n}\left(\frac{r_i\theta_i}{\delta_{1i}}\right)^{\delta_{1i}}\left(\frac{\theta_2}{\delta_{1n+1}}\right)^{\delta_{1n+1}}\left(\sum_{i=1}^{n+1}\delta_{1i}\right)^{\sum_{i=1}^{n+1}\delta_{1i}} \tag{2.9}$$

$$\text{subject to } \delta_{01}=1,\ -\alpha_i\delta_{01}+\delta_{1i}+\alpha_i\delta_{1n+1}=0, \forall i=1\ldots n.$$

Therefore the optimal solution is as follows $\delta_{01}^{*}=1,\ \delta_{1i}=\alpha_i\left(1-\delta_{1n+1}\right), \forall\ i=1\ldots n$. Now substituting $\delta_{01}^{*},\ \delta_{1i}$, for $i=1\ldots n.$ in model (2.9), the dual function is obtained as follows

$$d\left(\delta_{1n+1}\right) = \theta\prod_{i=1}^{n}\left(\frac{r_i\theta_i}{\alpha_i\left(1-\delta_{1n+1}\right)}\right)^{\alpha_i\left(1-\delta_{1n+1}\right)}\left(\frac{\theta_2}{\delta_{1n+1}}\right)^{\delta_{1n+1}}\left(\left(\sum_{i=1}^{n}\alpha_i\right)\left(1-\delta_{1n+1}\right)+\delta_{1n+1}\right)^{\left(\left(\sum_{i=1}^{n}\alpha_i\right)\left(1-\delta_{1n+1}\right)+\delta_{1n+1}\right)} \tag{2.10}$$

To find the optimal solution δ_{1n+1}, we have to maximize the dual function $d\left(\delta_{1n+1}\right)$. Taking logarithm on both sides of equation (2.10) and differentiating with respect to δ_{1n+1} and then equating to zero, we have

$$\frac{d}{d\delta_{1n+1}}\left[\log\left\{d\left(\delta_{1n+1}\right)\right\}\right]=0 \text{ i.e. } \log\left(\frac{\theta_2}{\delta_{1n+1}}\right)+\left(1-\sum_{i=1}^{n}\alpha_i\right)\log\left\{\left(\sum_{i=1}^{n}\alpha_i\right)\left(1-\delta_{1n+1}\right)+\delta_{1n+1}\right\}-\sum_{i=1}^{n}\alpha_i\log\left(\frac{r_i\theta_1}{\alpha_i\left(1-\delta_{1n+1}\right)}\right)=0$$

Again, since $0<\delta_{1n+1}<1$ as $\delta_{1i}>0, \forall i$

$$\frac{d^2}{d\delta_{1n+1}^2}\left[\log\left\{d\left(\delta_{1n+1}\right)\right\}\right] =-\frac{1}{\delta_{1n+1}}-\frac{\sum_{i=1}^{n}\alpha_i}{1-\delta_{1n+1}}+\frac{\left(1-\sum_{i=1}^{n}\alpha_i\right)^2}{\left(\sum_{i=1}^{n}\alpha_i\right)\left(1-\delta_{1n+1}\right)+\delta_{1n+1}}=-\left[\frac{\sum_{i=1}^{n}\alpha_i}{1-\delta_{1n+1}}+\left(\frac{\left(\sum_{i=1}^{n}\alpha_i\right)\left\{1+\delta_{1n+1}\left(1-\sum_{i=1}^{n}\alpha_i\right)\right\}}{\delta_{1n+1}\left\{\left(\sum_{i=1}^{n}\alpha_i\right)\left(1-\delta_{1n+1}\right)+\delta_{1n+1}\right\}}\right)\right]<0.$$

Method 2.5. Max-additive operator

Next we solve model (2.3) using max additive operator as follows

$$\begin{aligned}&\text{maximize } \mu_{\tilde{R}}\left(R\left(x_1\ldots x_n\right)\right)+\mu_{\tilde{C}}\left(C\left(x_1\ldots x_n\right)\right)\\&\text{subject to}\\&\mu_{\tilde{R}}\left(R\left(x_1\ldots x_n\right)\right),\mu_{\tilde{C}}\left(C\left(x_1\ldots x_n\right)\right)\in[0,1],\\&x_i>0,\ \text{for } i=1\ldots n\end{aligned}$$

i.e. we have

$$\begin{aligned}&\text{maximize } \frac{pa}{R_1-R_0}\prod_{i=1}^{n}x_i^{\alpha_i}-\frac{1}{c_0}\sum_{i=1}^{n}r_ix_i\\&\text{subject to } \ x_i>0, i=1\ldots n.\end{aligned}$$

Now if $\frac{pa}{R_1-R_0}\prod_{i=1}^{n}x_i^{\alpha_i}-\frac{1}{c_0}\sum_{i=1}^{n}r_ix_i \ge x_{n+1}$, our problem becomes

$$\begin{aligned}&\text{maximize} \quad x_{n+1}\\ &\text{subject to } \frac{pa}{R_1-R_0}\prod_{i=1}^{n}x_i^{\alpha_i}-\frac{1}{c_0}\sum_{i=1}^{n}r_ix_i \ge x_{n+1}\\ &\qquad x_i>0, \quad \text{for } i=1\ldots n.\end{aligned} \tag{2.11}$$

Rewriting model (2.11) as PGPP form, we have the model as follows

$$\begin{aligned}&\text{minimize} \quad x_{n+1}^{-1}\\ &\text{subject to } \frac{R_1-R_0}{c_0pa}\left(\prod_{i=1}^{n}x_i^{-\alpha_i}\right)\left(\sum_{i=1}^{n}r_ix_i\right)+\frac{R_1-R_0}{pa}x_{n+1}\prod_{i=1}^{n}x_i^{-\alpha_i}\le 1\\ &\qquad x_i>0, \quad \text{for } i=1\ldots n.\end{aligned} \tag{2.12}$$

Model (2.12) is a posynomial PGPP with DD being zero. Its DGPP is

$$\text{Maximize } d\left(\delta_{01},\delta_{11},\delta_{12}\ldots\delta_{1n+1}\right)=\left(\frac{1}{\delta_{01}}\right)^{\delta_{01}}\prod_{i=1}^{n}\left(\frac{r_i\left(R_1-R_0\right)}{c_0pa\delta_{1i}}\right)^{\delta_{1i}}\left(\frac{R_1-R}{pa\delta_{1n+1}}\right)^{\delta_{1n+1}}\left(\sum_{i=1}^{n+1}\delta_{1i}\right)^{\sum_{i=1}^{n+1}\delta_{1i}}$$

subject to

$$\delta_{01}=1,\ \left(1-\sum_{j=1}^{n}\alpha_j\right)\delta_{1i}-\alpha_i\delta_{1n+1}=0,\ \forall i=1\ldots n,\ -\delta_{01}+\delta_{1n+1}=0.$$

The optimal solution to the problem is $\delta_{01}^*=1,\ \delta_{1n+1}^*=1,\ \delta_{1i}^*=\frac{\alpha_i}{1-\sum_{i=1}^{n}\alpha_i}, i=1\ldots n.$

Therefore $d^*\left(\delta_{01}^*,\delta_{11}^*,\delta_{12}^*\ldots\delta_{1n+1}^*\right)=\frac{1}{1-\sum_{i=1}^{n}\alpha_i}\left(\frac{R_1-R_0}{pa}\right)^{\frac{1}{1-\sum_{i=1}^{n}\alpha_i}}\prod_{i=1}^{n}\left(\frac{r_i}{c_0\alpha_i}\right)^{\frac{\alpha_i}{1-\sum_{i=1}^{n}\alpha_i}}$. From primal dual relations, we have $x_{n+1}^{-1}=\delta_{01}^*d^*\left(\delta_{01}^*,\delta_{11}^*,\delta_{12}^*\ldots\delta_{1n+1}^*\right)$ and $\frac{R_1-R_0}{c_0pa}\left(\prod_{i=1}^{n}x_i^{-\alpha_i}\right)r_ix_i=\frac{\delta_{1i}^*}{\sum_{i=1}^{n+1}\delta_{1i}^*}, \forall i=1\ldots n$ and hence $\frac{R_1-R_0}{pa}x_{n+1}\prod_{i=1}^{n}x_i^{-\alpha_i}=\frac{\delta_{1n+1}^*}{\sum_{i=1}^{n+1}\delta_{1i}^*}$.

Here optimal inputs are $x_i^*=\left(\frac{\alpha_i}{r_i}\right)c_0^{\frac{1}{1-\sum_{i=1}^{n}\alpha_i}}\left(\frac{pa}{R_1-R_0}\right)^{\frac{1}{1-\sum_{i=1}^{n}\alpha_i}}\prod_{i=1}^{n}\left(\frac{\alpha_i}{r_i}\right)^{\frac{\alpha_i}{1-\sum_{i=1}^{n}\alpha_i}}$ for $i=1\ldots n$ and optimal revenue are $R^*\left(x_1^*\ldots x_n^*\right)=pa\prod_{i=1}^{n}\left(\frac{\alpha_i}{r_i}\right)^{\alpha_i}\left\{c_0^{\frac{1}{1-\sum_{i=1}^{n}\alpha_i}}\left(\frac{pa}{R_1-R_0}\right)^{\frac{1}{1-\sum_{i=1}^{n}\alpha_i}}\prod_{i=1}^{n}\left(\frac{\alpha_i}{r_i}\right)^{\frac{\alpha_i}{1-\sum_{i=1}^{n}\alpha_i}}\right\}^{\sum_{i=1}^{n}\alpha_i}$.

Method 2.6. Max-product operator

Next we solve model (2.3) using max product operator as follows

$$\begin{array}{l}\text{maximize } \mu_{\tilde{R}}\left(R\left(x_1 \ldots x_n\right)\right) . \mu_{\tilde{C}}\left(C\left(x_1 \ldots x_n\right)\right) \\ \text{subject to} \\ \mu_{\tilde{R}}\left(R\left(x_1 \ldots x_n\right)\right), \mu_{\tilde{C}}\left(C\left(x_1 \ldots x_n\right)\right) \in [0,1],\ x_i > 0, \text{ for } i = 1 \ldots n.\end{array}$$

i.e. we have

$$\begin{array}{l}\text{Maximize } \dfrac{\prod_{i=1}^{n} pa x_i^{\alpha_i} - R_0}{R_1 - R_0} . \dfrac{c + c_0 - \sum_{i=1}^{n} r_i x_i}{c_0} \\ \text{subject to } \dfrac{\prod_{i=1}^{n} pa x_i^{\alpha_i} - R_0}{R_1 - R_0}, \dfrac{c + c_0 - \sum_{i=1}^{n} r_i x_i}{c_0} \in [0,1] \\ x_i > 0, \text{ for } i = 1 \ldots n.\end{array}$$

Let us assume that $\dfrac{\prod_{i=1}^{n} pa x_i^{\alpha_i} - R_0}{R_1 - R_0} \geq x_{n+1}$, $\dfrac{c + c_0 - \sum_{i=1}^{n} r_i x_i}{c_0} \geq x_{n+2}$. Therefore the model becomes

$$\begin{array}{l}\text{Maximize } x_{n+1} . x_{n+2} \\ \text{subject to} \\ \dfrac{\prod_{i=1}^{n} pa x_i^{\alpha_i} - R_0}{R_1 - R_0} \geq x_{n+1}, \\ \dfrac{c + c_0 - \sum_{i=1}^{n} r_i x_i}{c_0} \geq x_{n+2}, \\ x_i > 0, \text{ for } i = 1 \ldots n+2.\end{array} \tag{2.13}$$

Equation (2.13) can be written in PGPP form as follows

$$\begin{array}{l}\text{minimize } x_{n+1}^{-1} x_{n+2}^{-1} \\ \text{subject to} \dfrac{1}{c + c_0} \sum_{i=1}^{n} r_i x_i + \dfrac{c_0}{c + c_0} x_{n+2} \leq 1, \quad \dfrac{R_0}{pa} \prod_{i=1}^{n} x_i^{-\alpha_i} + \dfrac{R_1 - R_0}{pa} x_{n+1} \prod_{i=1}^{n} x_i^{-\alpha_i} \leq 1\end{array} \tag{2.14}$$

which is a posynomial PGPP with DD being unity. Therefore its DGPP is as follows

Maximize $d\left(\delta_{01},\delta_{11},\delta_{12}\ldots\delta_{1n+1},\delta_{21},\delta_{22}\right)$ (2.15)

$$=\left(\frac{1}{\delta_{01}}\right)^{\delta_{01}}\prod_{i=1}^{n}\left(\frac{r_i}{\left(c+c_0\right)\delta_{1i}}\right)^{\delta_{1i}}\left(\frac{c_0}{\left(c+c_0\right)\delta_{1n+1}}\right)^{\delta_{1n+1}}$$

$$\left(\frac{R_0}{pa\delta_{21}}\right)^{\delta_{21}}\left(\frac{R_1-R_0}{pa\delta_{22}}\right)^{\delta_{22}}\left(\sum_{i=1}^{n+1}\delta_{1i}\right)^{\sum_{i=1}^{n+1}\delta_{1i}}\left(\sum_{i=1}^{2}\delta_{2i}\right)^{\sum_{i=1}^{2}\delta_{2i}}$$

subject to

$$\delta_{01}=1,\ \delta_{1i}-\alpha_i\delta_{21}-\alpha_i\delta_{22}=0,\ \forall\ i=1\ldots n,-\delta_{01}+\delta_{22}=0,\ -\delta_{01}+\delta_{1n+1}=0.$$

The optimal solution of this model (2.15) is obtained as $\delta_{01}^*=1,\ \delta_{22}^*=1,\ \delta_{1n+1}^*=1,$ $\delta_{1i}=\alpha_i\left(\delta_{21}+1\right),\forall\ i=1\ldots n.$ Substituting $\delta_{01}^*,\ \delta_{22}^*,\ \delta_{1n+1}^*,\ \delta_{1i}$ for $i=1\ldots n,$ in (2.15), the dual function is obtained as follows

$$d\left(\delta_{21}\right)=\prod_{i=1}^{n}\left(\frac{r_i}{\left(c+c_0\right)\alpha_i(\delta_{21}+1)}\right)^{\alpha_i(\delta_{21}+1)}\left(\frac{c_0}{c+c_0}\right)\left(\frac{R_0}{pa\delta_{21}}\right)^{\delta_{21}}\left(\frac{R_1-R_0}{pa}\right)$$ (2.16)

$$\left\{1+(\delta_{21}+1)\sum_{i=1}^{n}\alpha_i\right\}^{\left\{1+(\delta_{21}+1)\sum_{i=1}^{n}\alpha_i\right\}}(\delta_{21}+1)^{(\delta_{21}+1)}$$

Next we have to find the optimal value of δ_{21}, so we maximize dual function $d(\delta_{21})$. Taking logarithm on both sides of model (2.16), then differentiating with respect to δ_{21} and equating to zero, we get as follows

$$\frac{d}{d\delta_{21}}\{\ln(d(\delta_{21}))\}=0 \text{ i.e. } \sum_{i=1}^{n}\alpha_i\ln\frac{r_i}{a_i(c+c_0)(\delta_{21}+1)}+\ln\left(\frac{R_0}{pa\delta_{21}}\right)+\left(\sum_{i=1}^{n}\alpha_i\right)\ln\left\{1+(\delta_{21}+1)\sum_{i=1}^{n}\alpha_i\right\}+\ln\left(\delta_{21}+1\right)=0$$

Then we show that second order derivative is always negative i.e. we get

$$\frac{d^2}{d\delta_{21}^2}\left[\ln\left\{d\left(\delta_{21}\right)\right\}\right]=-\frac{\sum_{i=1}^{n}\alpha_i}{(\delta_{21}+1)}+\frac{1}{(\delta_{21}+1)}-\frac{1}{\delta_{21}}+\frac{\left(\sum_{i=1}^{n}\alpha_i\right)^2}{\left\{1+(\delta_{21}+1)\sum_{i=1}^{n}\alpha_i\right\}}=-\frac{1}{\delta_{21}(\delta_{21}+1)}-\frac{\sum_{i=1}^{n}\alpha_i}{(\delta_{21}+1)\left(1+(\delta_{21}+1)\sum_{i=1}^{n}\alpha_i\right)}<0.$$

3. Numerical examples

We now consider numerical examples to apply those fuzzy optimization techniques to solve Cobb-Douglas based firm production model in fuzzy environment one by one. The input data are given in Table 1, analogous to Creese (2010). And the outcome using crisp optimization technique to solve Cobb-Douglas based firm production model are given in Table 2.

Table 1 Input data of Cobb-Douglas based firm production model

No. of Inputs	Output elasticities of the Input components			Prices of the input components			Selling price of a unit product	Total productivity	Available cost
n	α_1	α_2	α_3	r_1	r_2	r_3	p	a	c
3	0.1	0.3	0.2	20	24	30	20	40	8500

Table 2 Output data using crisp optimization technique

Dual Variables				Primal Variables			Revenue
δ_{01}^*	δ_{11}^*	δ_{12}^*	δ_{13}^*	x_1^*	x_2^*	x_3^*	R^*
1.0	0.1	0.3	0.2	70.8333	177.0833	94.4444	14374.82

Suppose the input data in fuzzy environment is given in table 3.

Table 3 Input data of Cobb-Douglas based firm production model in fuzzy environment

No. of Inputs	Output elasticities of the Input components			Prices of the input components (Rs.)			Selling Price of a unit product (Rs.)	Total productivity	Available cost (Rs.)	Available tolerance (Rs.)
n	α_1	α_2	α_3	r_1	r_2	r_3	p	a	c	c_0
3	0.1	0.3	0.2	20	24	30	20	40	8500	300

On solving this model by Verdegay's approach (1982) in fuzzy environment, output data corresponding to different values of aspiration level β are obtained as given in Table 4.

Table 4 Output data of Cobb-Douglas based firm production model by Verdegay's approach

Aspiration Level	Dual Variables			Primal Variables			Cost (Rs.)	Revenue (Rs.)
β	δ_{01}^*	δ_{02}^*	δ_{03}^*	x_1^*	x_2^*	x_3^*	C^*	R^*
0.0	0.1	0.3	0.2	73.3333	183.3333	97.7778	8800	14677.11
0.1				73.0833	182.7083	97.4444	8770	14647.07
0.2				72.8333	182.0833	97.1111	8740	14616.99
0.3				72.5833	181.4583	96.7778	8710	14586.86
0.4				72.3333	180.8333	96.4444	8680	14556.70
0.5				72.0833	180.2083	96.1111	8650	14526.49
0.6				71.8333	179.5833	95.7778	8620	14496.24
0.7				71.5833	178.9583	95.4444	8590	14465.95
0.8				71.3333	178.3333	95.1111	8560	14435.61
0.9				71.0833	177.7083	94.7778	8530	14405.24
1.0				70.8333	177.0833	94.4444	8500	14374.82

Again by applying max-min operator by Zimmermann (1975), the output data is obtained as given in Table 5.

Table 5 Output data using Zimmermann's approach

Dual Variables	Primal Variables	Optimal Revenue	Optimal Cost	Aspiration level

δ_{01}^*	δ_{11}^*	δ_{12}^*	δ_{13}^*	δ_{14}^*	δ_{21}^*	δ_{22}^*	x_1^*	x_2^*	x_3^*	R^*	C^*	$\mu_{\tilde{R}}^*(R(x_1,\dots,x_n))$ and $\mu_{\tilde{C}}^*(C(x_1,\dots,x_n))$
1.0	4.797	14.391	9.594	0.5	47.470	0.5	72.081	180.202	96.108	14526.19	8649.71	0.5 and 0.5

Again by applying Sakawa's method (2013), the output data is obtained as given in Table 6.

Table 6 Output data using Sakawa's method

Dual Variables					Primal Variables			Optimal Revenue	Optimal Cost	Aspiration level
δ_{01}^*	δ_{11}^*	δ_{12}^*	δ_{13}^*	δ_{14}^*	x_1^*	x_2^*	x_3^*	R^*	C^*	$\mu_{\tilde{R}}^*(R(x_1,\dots,x_n))$ and $\mu_{\tilde{C}}^*(C(x_1,\dots,x_n))$
1.00	0.050	0.150	0.100	0.499	72.254	180.634	96.338	14547.07	8670.44	0.57 and 0.43

Again by applying max additive operator, the output data is obtained as given in Table 7.

Table 7 Output data using max additive operator

Dual variables					Primal variables			Optimal Revenue (Rs.)	Optimal Cost (Rs.)	Aspiration level
$\delta_{01}{}^*$	$\delta_{11}{}^*$	$\delta_{12}{}^*$	$\delta_{13}{}^*$	$\delta_{14}{}^*$	$x_1{}^*$	$x_2{}^*$	$x_3{}^*$	R^*	C^*	$\mu_{\tilde{R}}^*(R(x_1,\dots,x_n))$ and $\mu_{\tilde{C}}^*(C(x_1,\dots,x_n))$
1.00	0.25	0.75	0.50	1.00	72.080	180.201	96.107	14526.12	8649.63	0.5 and 0.5

Again by applying max product operator, the output data is obtained as given in Table 8.

Table 8 Output data using max product operator

Dual Variables							Primal variables			Optimal Revenue (Rs.)	Optimal Cost (Rs.)	Aspiration Level
$\delta_{01}{}^*$	$\delta_{11}{}^*$	$\delta_{12}{}^*$	$\delta_{13}{}^*$	$\delta_{14}{}^*$	$\delta_{21}{}^*$	$\delta_{22}{}^*$	$x_1{}^*$	$x_2{}^*$	$x_3{}^*$	R^*	C^*	$\mu_{\tilde{R}}^*(R(x_1,\dots,x_n))$ and $\mu_{\tilde{C}}^*(C(x_1,\dots,x_n))$
1.00	9.59	28.7	19.1	1.00	94.9	1.00	72.08	180.20	96.10	14526.22	8649.73	0.5 and 0.5

We now compare the results obtained by using different fuzzy optimization techniques to solve Cobb-Douglas based firm production model in fuzzy environment.

Table 9 Comparison of outcomes in different techniques

Method	Optimal Inputs			Optimal Revenue	Optimal Cost
	$x_1{}^*$	$x_2{}^*$	$x_3{}^*$	R^*	C^*
Zimmermann's approach	72.081	180.202	96.108	14526.19	8649.71

(max-min operator)					
Sakawa's method (max-min operator)	72.254	180.634	96.338	14547.07	8670.44
max-additive operator	72.080	180.201	96.107	14526.12	8649.63
max-product operator	72.081	180.203	96.108	14526.22	8649.73

Hence the optimal revenue in classical optimization technique is Rs. 14374.82 with optimal cost being Rs. 8500. But if same model is considered under fuzzy environment and solved by using max-min operator in Zimmermann's technique, the optimal revenue comes as Rs. 14526.19 with optimal cost being Rs. 8649.71. As maximizing revenue is primary objective to decision maker, so this outcome is more acceptable than the solution under crisp environment.

Again if max-min operator in Sakawa's technique is used to solve the same model under fuzzy environment, the optimal revenue comes is Rs. 14547.07, a far more acceptable solution than the solution under crisp environment.

Again if max-additive operator is used to solve the same model under fuzzy environment, the optimal revenue comes is Rs. 14526.12, another better optimal solution than crisp solution.

Again if max-product operator is used to solve the same model under fuzzy environment, the optimal revenue comes is Rs. 14526.22, again one better optimal solution than crisp solution.

4. Conclusion

In this paper, we have considered Cobb-Douglas based firm production model in fuzzy environment. In this model, the firm behaves so as to maximize the revenue under limited total expenditure cost. Also its aim is to minimize the total expenditure costs subject to target revenue. To match with reality, here the model is considered under fuzzy environment and solved using different fuzzy optimization techniques.

In this paper, geometric programming technique is applied to solve the model in all these fuzzy techniques. The advantage of geometric programming over other optimization techniques is that it provides us with a systematic approach for solving a class of non-linear optimization problems by finding the optimal value of objective function and then the optimal values of decision variables are obtained. Moreover this method often reduces a complex optimization problem to a set of simultaneous equations.

We know that decision maker is the king and his decision is final. So our proposal is to collect information from decision maker first, and then based on such information, fuzzy optimization approach be chosen. Then geometric programming technique is used to find optimal solution. Next, we present the optimal solution to decision maker. If he is satisfied with this solution, stop. Otherwise another fuzzy technique needs to be used until decision maker is satisfied.

We are in the process to develop few interesting results on Cobb-Douglas based firm production model in fuzzy environment. We also plan to use proposed technique to generate optimal solutions in agro-industrial sector in near future.

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